CONSTRUCTIONS USING A COMPASS AND TWICE-NOTCHED STRAIGHTEDGE

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1. Introduction

It is impossible to trisect an arbitrary angle. So mathematicians have claimed, with confidence, for more than 160 years. The statement is provocative. To a mathematician, the statement embodies the beauty of algebra and its applications to geometry, hints at Galois theory, and is a rare example of a statement of the nonexistence of a solution. To recreational mathematicians, it is often thought of as a challenge. Every year, mathematicians around the world receive letters from the general population making claims to the contrary. Their solutions fall into two main categories: they are either false or do not adhere to the rules of constructions.

In our provocative statement, we often omit the qualifying phrase *using only a straightedge and compass.* This is a restriction whose popularity is most probably due to the writings of Plato (ca. 427 – 347 B.C.)[6]. But according to Pappus (late 3rd century, A.D.), the ancient Greeks (ancient already to him) classified problems in geometric construction into three types. A problem is called *plane* if it can be solved using only a straightedge and compass; it is called *solid* if it can be solved using one or more conic section(s); and it is called *linear* if the solution requires a more complicated curve. In particular, the ancient Greeks had already found solid solutions to the trisection problem, as well as to the problem of doubling the cube. They suspected that neither problem was plane, a fact that was finally established by Pierre L. Wantzel (1814 – 1848) in 1837 (though some have argued that Gauss must have known how to do this soon after writing *Disquisitiones* in 1798 [4, 5]).

I find this classification very intriguing, for it reflects a point of view that is appealing to modern algebraic geometers. The ancient Greeks somehow observed that the simplest problems are those that can be solved using quadratics, and that the next simplest class is the class of problems solvable with quadratics and cubics. Even their terminology (with the exception of "linear") is very appropriate. The terms "plane" and "solid" are meant to suggest the two-dimensional and three-dimensional natures of the solutions. Thus, a plane construction should involve equations of degree two, and a solid construction should involve equations of degree three. The modern mathematician might be tempted to partition the linear problems (a poor choice of terminology) further into algebraic and transcendental problems, but we would probably do little more.

I also find it intriguing that this classification scheme does not obviously include the following trisection algorithm, due to Archimedes (287 – 212 B.C.).

Theorem 1.1 (Archimedes' Trisection Algorithm)**.** *If we are in possession of a compass and a straightedge that is notched in two places, then it is possible to trisect an arbitrary angle.*

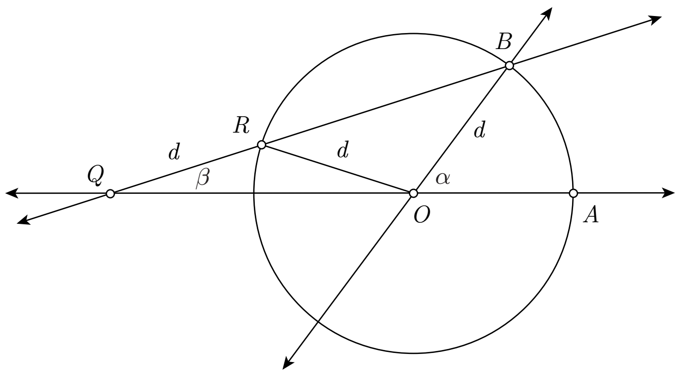

FIGURE 1.

Proof. We show how to trisect an acute angle. This is enough, since we know it is possible to trisect a right angle using only a straightedge (without notches) and compass, and an obtuse angle is the sum of a right angle and an acute angle. So, given an acute angle α with vertex O, construct the circle centered at O and with radius d, where d is the distance between the two notches on our straightedge. Let $\alpha = \angle AOB$ for points A and B on this circle, as in Figure 1. Draw the line OA. Arrange the straightedge so that one notch is on the circle, the other notch is on the line OA on the opposite side of O from A, and the straightedge goes through the point B (this is the step that does not adhere to the plane rules of constructions). Let the points at the notches be Q and R, and let $\beta = \angle RQO$. The triangle ΔRQO is isosceles, since $|RQ| = |RO| = d$. Thus $\angle ROQ = \beta$, hence $\angle BRO = 2\beta$. Since ΔROB is also isosceles, $\angle RBO = \angle BRO = 2\beta$, so $\angle ROB = \pi - 4\beta$. Summing the angles at O, we get

$$\beta + (\pi - 4\beta) + \alpha = \pi,$$

which implies that

$$\alpha = 3\beta.$$

Thus, we have trisected the angle α. □

Discovering how this algorithm fits into the classification scheme is exactly the sort of analysis a modern algebraic geometer would attempt in order to understand what other problems can be solved using a twice-notched straightedge. In this paper, we will briefly study solid constructions and show that a point has a solid construction if and only if it lies in a 2-3-tower over $\mathbb{Q}$. That is, the point (x, y) has a solid construction if and only if, when represented as a complex number $x + iy$, it lies in a subfield K of $\mathbb{C}$ for which there exists a finite sequence of subfields $K_0, K_1, ..., K_n$ satisfying

$$\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n = K \tag{1}$$

and the index $[K_j : K_{j-1}]$ at each step is 2 or 3. This result is not new (see Videla [14]). We will then study constructions using a compass and twice-notched straightedge. With the aid of Theorem 1.1 and a result due to Nicomedes, we will demonstrate that every point that has a solid construction can be constructed using a compass and twice-notched straightedge. Thus, the twice-notched straightedge is at least as powerful a tool as a conic drawing tool. We will demonstrate that every point constructible with a compass and twice-notched straightedge lies in a tower of fields over $\mathbb{Q}$ (a sequence of nested fields as in (1)) in which the degree of the extension at each step is 2, 3, 5, or 6. Consequently, it is not possible to construct a regular 23-gon or 29-gon using only a compass and twice-notched straightedge. Finally, we will employ these tools to construct several points whose x-coordinates are roots of a quintic that is not solvable by radicals.

2. Plane Constructions

Let us begin by reviewing briefly the rules for constructions that use only a straightedge and a compass. A more detailed treatment appears in [2].

We start with two points O and P, which we declare constructible. Given two constructible points A and B, we can construct the line through A and B, and the circle centered at A that passes through B. The points where (distinct) constructed lines and circles intersect are called *constructible points*.

The classical algebraic analysis of plane constructions goes as follows. We put a Cartesian coordinate system on our plane, choosing O to be the origin and P to be the point $(1, 0)$. A point (x, y) in the plane can be represented by the complex number $x + iy$. A complex number $x + iy$ is called a *constructible number* if the point (x, y) is a constructible point. It is not difficult to verify that, if A and B are constructible numbers, then so are $A + B$, $-A$, AB, and $1/A$ (for $A \neq O$). As a result, the set of constructible numbers forms a field (sometimes called the *surd field*).

A *construction* $\mathcal{C}$ is a finite set of points $\mathcal{C} = \{O, P, A_1, ..., A_n\}$ such that A_{k+1} is a point of intersection of lines and/or circles constructed from the points in the subconstruction $\mathcal{C}_k = \{O, P, A_1, ..., A_k\}$. For a construction $\mathcal{C}$, let us define $K[\mathcal{C}]$ to be the smallest field containing $\mathcal{C}$ and i that is closed under complex conjugation. It is not too difficult to verify that $K[\mathcal{C}_k][A_{k+1}]$ is of degree 1 or 2 over $K[\mathcal{C}_k]$. Note that $\mathcal{C}_0 = \{O, P\}$ and $K[\mathcal{C}_0] = \mathbb{Q}[i]$, which is of degree 2 over $\mathbb{Q}$. Thus, every constructible number lies in a field that is in a 2-tower over $\mathbb{Q}$.

Furthermore, since we can bisect angles and find the square roots of lengths, we can solve general quadratic equations, so every number that lies in a 2-tower over $\mathbb{Q}$ is, in fact, constructible.

Though we often associate this result with Galois theory, all that is required for its proof is an understanding of fields and degrees of field extensions, concepts that were articulated by Niels Henrik Abel (1802 – 1829). Though Evariste Galois (1811 – 1832) died only a short time after Abel and well before the publication of Wantzel's paper, Wantzel would not have been aware of Galois's work because the bulk of it was not published until 1846.

3. Solid Constructions

In solid constructions, we allow for the use of a (possibly only hypothetical) conic drawing tool. Given a constructible point A, a constructible line l, and a constructible length e, we are permitted to draw the conic section with focus A, directrix l, and eccentricity e. The constructible points in this context are then the points where a pair of (distinct) constructible lines, circles, or conics intersect.

In particular, we can draw the parabola $y = x^2$. Consider the circle that is centered at (a, b) and goes through the origin. This circle has the equation

$$(x-a)^2 + (y-b)^2 = a^2 + b^2.$$

If a point $(x, y) \neq (0, 0)$ also lies on the parabola $y = x^2$, then

$$\begin{aligned} (x-a)^2 + (x^2-b)^2 &= a^2 + b^2, \\ x^2 - 2ax + x^4 - 2bx^2 &= 0, \\ x^3 + (1-2b)x - 2a &= 0. \end{aligned} \tag{2}$$

Thus, given real values a and b, we can construct the real roots of the cubic in (2). In particular, given a length r, we can find $\sqrt[3]{r}$ by setting $a = -r/2$ and $b = 1/2$. Hence, we can double the cube.

Trisecting angles is a little more complicated. We can trisect a given angle α if we can construct $x = e^{i\alpha/3}$, which satisfies the equation

$$x^3 - e^{i\alpha} = 0.$$

Note that

$$x^{-3} = e^{-i\alpha},$$

which gives

$$x^3 + x^{-3} = 2\cos\alpha,$$

a quantity that is constructible using plane tools. Let $\omega = x + x^{-1} = 2\cos(\alpha/3)$. Then

$$\begin{aligned} \omega^3 &= x^3 + 3x + 3x^{-1} + x^{-3} \\ &= 3\omega + 2\cos\alpha, \end{aligned}$$

whence

$$\omega^3 - 3\omega - 2\cos\alpha = 0. \tag{3}$$

Since this equation has real constructible coefficients, we can construct the real roots ω of (3) and use the appropriate root to construct the angle $\alpha/3$.

Since we can trisect angles and find the cube roots of lengths, we can find the cube root of any constructible complex number. In light of Cardano's and Ferrari's formulas expressing the solutions to cubics and quartics in terms of radicals[13], we can construct the roots of any cubic (or quartic) equation with constructible coefficients. Hence, any number in a 2-3-tower over $\mathbb{Q}$ has a solid construction.

Conversely, suppose we have a solid construction $\mathcal{C} = \{O, P, A_1, ..., A_n\}$. Then the point A_{k+1} is a point of intersection of lines, circles, or conic sections created from the points in $\mathcal{C}_k$. That is, A_{k+1} is a point of intersection of lines, circles, or conic sections described by equations with coefficients in $K[\mathcal{C}_k]$. Since two conic sections intersect in at most four points, it is not surprising that the point A_{k+1} is a root of a (possibly reducible) quartic polynomial in $K[\mathcal{C}_k][x]$, a fact that is easy enough to verify (see Videla[14] for details). As a consequence, A_{k+1} is in a 2-3-tower over $K[\mathcal{C}_k]$ (of degree at most 12), and it follows that any point with a solid construction lies in 2-3-tower over $\mathbb{Q}$.

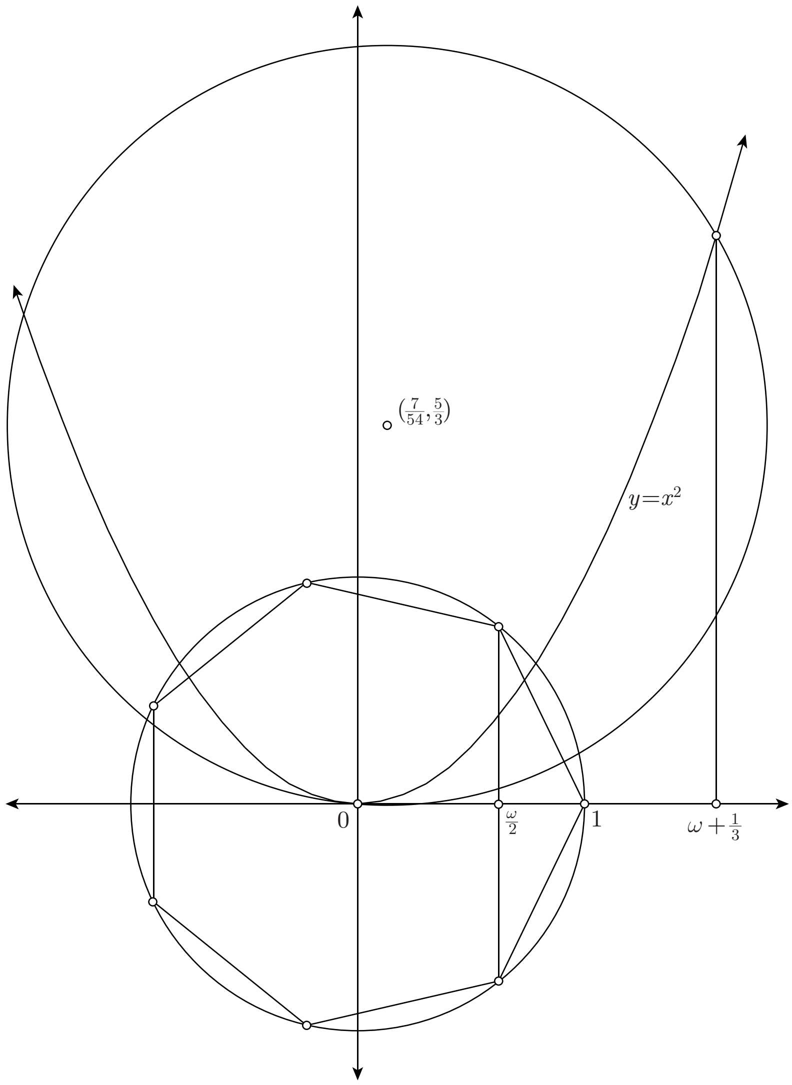

FIGURE 2. A solid construction of the regular 7-gon. We construct the circle centered at $\left(\frac{7}{54}, \frac{5}{3}\right)$ that goes through the origin. We drop a perpendicular from a point of intersection of this circle with the parabola $y = x^2$, to find the point $(\omega + 1/3, 0)$. We find the point $(\omega/2, 0)$ and the perpendicular to the x-axis through this point. This perpendicular intersects the unit circle at two of the seven points of a regular 7-gon. We use those points to find the rest of the vertices.

Example. The regular 7-gon has a solid construction, a fact that was known to Archimedes [8]. To construct it, we must construct $x = e^{2\pi i/7}$, which satisfies the equation

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0.$$

Let $\omega = x + x^{-1} = 2\cos(2\pi/7)$. Then

$$\begin{aligned}\omega^3 &= x^3 + 3x + 3x^{-1} + x^{-3},\\ \omega^2 &= x^2 + 2 + x^{-1},\end{aligned}$$

so

$$\omega^3 + \omega^2 - 2\omega - 1 = 0.$$

To exploit (2), we complete the cube. That is, we make the substitution $z = \omega + 1/3$, so that

$$(z - 1/3)^3 + (z - 1/3)^2 - 2(z - 1/3) - 1 = 0,$$

which simplifies to

$$z^3 - \frac{7}{3}z - \frac{7}{27} = 0.$$

We therefore choose $(a, b) = (7/54, 5/3)$ and proceed as indicated in Figure 2.

4. Twice-notched Straightedge Constructions

The classification of constructions is based on the intersection of curves. Though constructions using a twice-notched straightedge do not in any obvious way involve the intersection of curves, with a little thought we can find an appropriate interpretation. For example, in Archimedes' trisection algorithm, suppose that we do not restrict Q to lie on the line OA, but instead allow it to trace out a curve as the point R moves around the circle in such a manner that the line QR always passes through B.

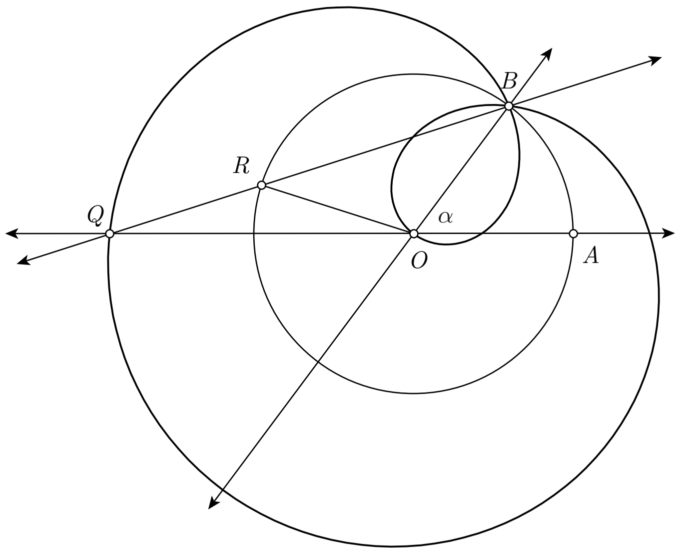

Figure 3. The limaçon interpretation of Archimedes' trisection algorithm.

Let us change our vantage point. Let B be the origin, let $d = 1$, and let the circle have the equation $r = 2\cos\theta$ in polar coordinates. Then Q is a point at a distance 1 away from a point R on the circle, so traces out the curve with polar equation

$$r = 1 + 2\cos\theta.$$

This curve is known as a *limaçon*, a variation on the cardioid and a frequent guest in calculus texts (see Figure 3).

Alternatively, we can let Q move along the line OA and consider the curve traced out by R. Again, if we choose the origin at B and rotate the axes so that the line has the equation $r = a\sec\theta$ for $a = \sin\alpha$, then R traces out the curve described by

$$r = a\sec\theta - 1.$$

This curve is known as a *conchoid* (see Figure 4). The conchoid, sometimes referred

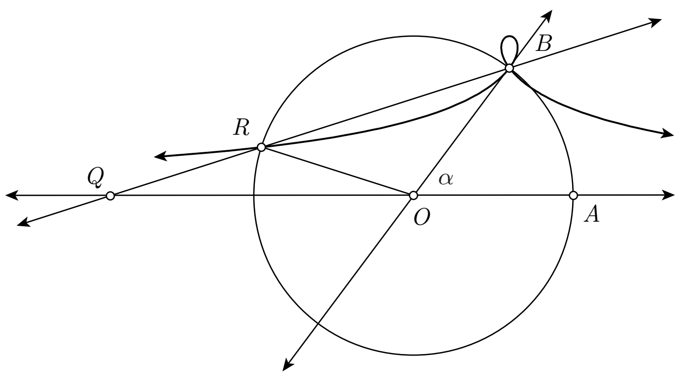

FIGURE 4. The conchoid interpretation of Archimedes' trisection algorithm.

to as the *conchoid of Nicomedes*, was studied extensively by Nicomedes (ca. 250 B.C.), who used it to double the cube and, more generally, find cube roots.

Theorem 4.1 (Nicomedes). *Given a constructible length a, it is possible to find $\sqrt[3]{a}$ using a compass and twice-notched straightedge.*

Proof. We first describe the procedure assuming that the notches are a unit distance apart. Note that it is enough to find $\sqrt[3]{a}$ for $a < 1$ (since we can construct inverses). We begin by constructing a rectangle $OBCD$ with dimensions $2a$ and 2, as in Figure 5, where $|OP| = 1$ and $|OA| = a$. The point E is constructed so that EA is perpendicular to OA and $|EB| = 1$. The point F is constructed so that it is a distance $2a$ away from O. We then construct the line l through B that is parallel to EF. All these points are constructible using a compass and straightedge. For our nonplane construction, we place one notch on the line l, the other notch on the line OA, and adjust these points so that the straightedge passes through the point E, giving the points G and H. In the language of curves, in this last step we construct the conchoid $r = \sec\theta + 1$ using the coordinate system centered at E with x-axis along the line EB. This conchoid intersects OA at H. We claim that the distance $x = |BH|$ is $2\sqrt[3]{a}$.

To analyze the algorithm, let us also draw the line HC that intersects OP at J, and let $y = |EG|$. We begin by applying the Pythagorean theorem to triangles

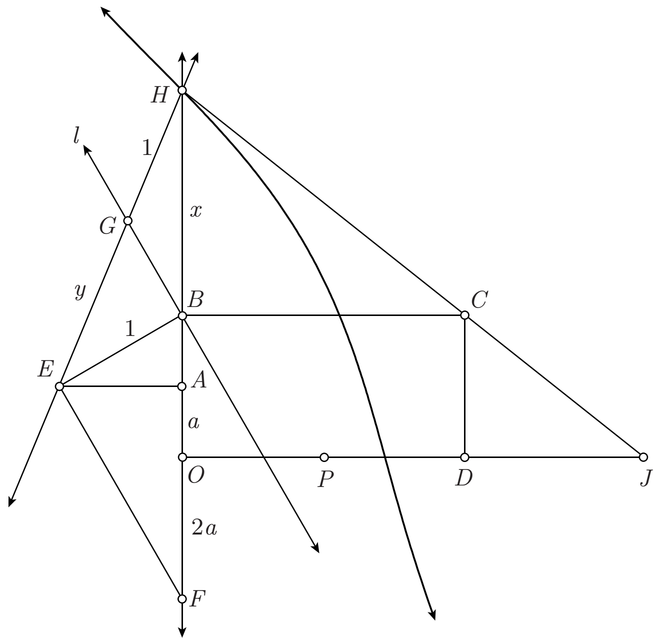

FIGURE 5. Nicomedes' algorithm for finding cube roots (see Theorem 4.1).

ΔAEB and ΔAEH:

$$\begin{aligned} a^2 + |AE|^2 &= 1, \\ (x+a)^2 + |AE|^2 &= (1+y)^2, \\ x(x+2a) &= y(y+2), \\ \frac{x}{y} &= \frac{y+2}{x+2a}. \qquad (4) \end{aligned}$$

Since l is parallel to the side EF in ΔHEF, we know that

$$\frac{y}{1} = \frac{4a}{x}.$$

Since ΔHBC is similar to ΔCDJ, we also know that

$$\frac{x}{2a} = \frac{2}{|DJ|},$$

so $|DJ| = 4a/x = y$. Because ΔHOJ is similar to ΔCDJ, we get

$$\begin{aligned} \frac{x+2a}{2a} &= \frac{y+2}{y}, \\ \frac{y+2}{x+2a} &= \frac{y}{2a}, \end{aligned}$$

which combined with (4) gives

$$\frac{y}{2a} = \frac{x}{y}.$$

Substituting $y = 4a/x$, we obtain

$$\frac{4a}{2ax} = \frac{x^2}{4a},$$

which implies that $x^3 = 8a$, so $\sqrt[3]{a} = x/2$, as claimed.

Finally, if the notches are a distance d apart, scale Figure 5 by d. Then $x = 2d\sqrt[3]{a}$, from which we can isolate $\sqrt[3]{a}$ using plane rules. □

As a corollary of the combined results of Archimedes and Nicomedes, we get the following result:

Theorem 4.2. *Every point that has a solid construction can be constructed using a compass and twice-notched straightedge.*

In other words, the twice-notched straightedge is at least as powerful a tool as a conic drawing tool. In Section 6, we will show that it is, in fact, a superior tool. To be more precise, there are some points that are constructible using a compass and twice-notched straightedge but do not have solid constructions.

5. Limitations of constructions using a compass and twice-notched straightedge

Let us first agree on the rules for constructions using a twice-notched straightedge. We start with a point O and use our twice-notched straight edge to draw a point P whose distance from O is the same as the distance between the notches. Hence, the distance between the notches is 1. In plane constructions, constructible points are the points of intersection of constructible lines and circles, which we can draw. The way we are using a twice-notched straightedge, we can construct the point(s) of intersection of a conchoid or limaçon with a circle or line, but we cannot actually draw these exotic curves. In particular, we cannot (in general) find the points of intersection of two of these curves.

To further analyze the types of points that we can construct, we will need to know the equations (in rectangular coordinates) of the conchoid and limaçon. We begin with the conchoid.

The conchoid is created using a constructible point and line. Without loss of generality, we may assume that the point is the origin O and the line is a vertical line $x = a$, which has the polar equation $r = a\sec\theta$. If we place one notch on the given line and make the straightedge pass through O, then as the designated notch traverses the line, the other notch traces out one of the two curves whose polar equations are

$$r = a\sec\theta \pm 1.$$

The choice of plus or minus depends on whether or not the line is between the second notch and the origin. Converting to rectangular coordinates, we get

$$\begin{aligned} r\cos\theta &= a \pm \cos\theta, \\ x - a &= \pm\cos\theta, \\ (x-a)r &= \pm x, \\ (x-a)^2 r^2 &= x^2, \\ (x-a)^2(x^2+y^2) &= x^2. \end{aligned} \tag{5}$$

The curve described by (5) has degree four, so when we intersect it with a line, we arrive at points whose x-coordinates are roots of quartic polynomials. This yields nothing new, for quartics are solvable using square and cube roots.

When we intersect this curve with a circle, Bezout's theorem tells us to expect up to eight points of intersection. However, a general circle has the equation

$$(x-b)^2 + (y-c)^2 = s^2,$$

so

$$x^2 + y^2 = s^2 + 2bx + 2cy - b^2 - c^2. \tag{6}$$

Thus, when we intersect this circle with the curve given by (5), we get the equation

$$(x-a)^2(s^2 + 2bx + 2cy - b^2 - c^2) = x^2, \tag{7}$$

which now has degree three. Hence, we get at most six points of intersection. Algebraic geometers will recognize that, of the eight points of intersection guaranteed by Bezout's theorem (over $\mathbb{P}^2(\mathbb{C})$), two are at infinity.

Note that (7) is linear in y and that it can be rewritten as an equation of the form

$$A(x)y = B(x),$$

where $A(x)$ and $B(x)$ are of degree two and three, respectively. This leads to

$$\begin{aligned} A(x)(y-c) &= B(x) - cA(x), \\ A^2(x)(y-c)^2 &= (B(x) - cA(x))^2, \\ A^2(x)(s^2 - (x-b)^2) - (B(x) - cA(x))^2 &= 0. \end{aligned}$$

The polynomial in this last equation has degree six. Thus, if a, b, and c belong to a field K and (x, y) is a point of intersection of the pair of conchoids and circle described in (5) and (6), then x belongs to a field extension of degree at most six over K and y belongs to a field of degree at most two over that.

The exotic curves that are produced using a point and circle are variations on the limaçon. Again, without loss of generality, we may assume that the point is the origin O and that the center of the circle is on the x-axis. We can therefore represent the circle with the equation

$$(x-a)^2 + y^2 = t^2,$$

which has the polar equation

$$r^2 - 2ar\cos\theta + a^2 = t^2.$$

If we make the straightedge go through O while a notch traverses the given circle, then the other notch traces out one of the two paths with polar description

$$(r \pm 1)^2 - 2a(r \pm 1)\cos\theta + a^2 = t^2.$$

(Examples of such paths are shown in Figure 6.) We manipulate this expression as follows:

$$\begin{aligned} r^2 \pm 2r + 1 - 2ar\cos\theta \mp 2a\cos\theta + a^2 &= t^2, \\ r^2 + 1 - 2ar\cos\theta + a^2 - t^2 &= \pm(2a\cos\theta - 2r), \\ (r^2 + 1 - 2ar\cos\theta + a^2 - t^2)^2 r^2 &= 4(ar\cos\theta - r^2)^2. \end{aligned}$$

Converting to rectangular coordinates, we get

$$(x^2 + y^2 - 2ax + 1 + a^2 - t^2)^2(x^2 + y^2) = 4(x^2 + y^2 - ax)^2. \tag{8}$$

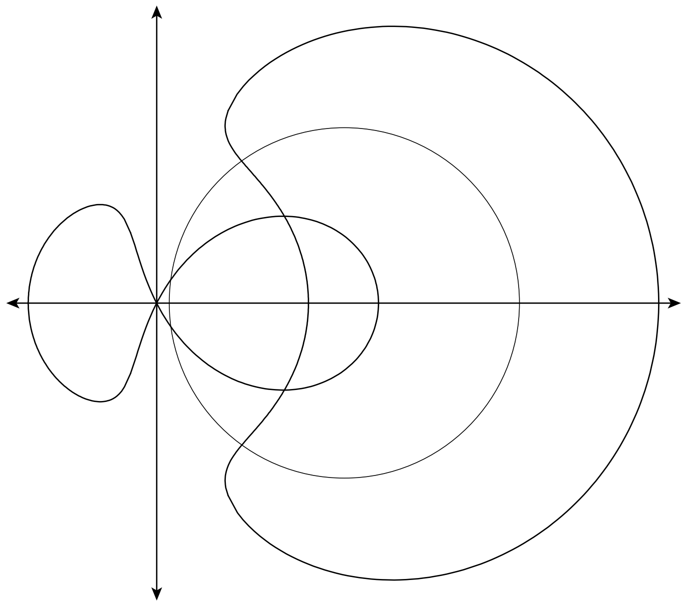

FIGURE 6. Generalizations of the limaçon – the curves traced out by a notch while keeping the other notch on the circle and making the straightedge go through the origin. In this example, the circle has radius $5/4$ and is centered at $(4/3, 0)$.

This equation is of degree six, so intersecting with a line, we expect at most six points. By Bezout's theorem, we expect its intersection with a circle to yield twelve points, but there are again a lot of points at infinity. Substituting (6) in the appropriate places in (8), we obtain

$$\begin{aligned}(s^2+2bx+2cy-b^2-c^2-2ax+1+a^2-t^2)^2(s^2+2bx+2cy-b^2-c^2)\\ = 4(s^2+2bx+2cy-b^2-c^2-ax)^2,\end{aligned} \tag{9}$$

which is now of degree three. Thus, the intersection of the generalized limaçon with a circle gives a polynomial in x of degree six.

This polynomial can be found explicitly. We note that (9) is cubic in y, so the problem is a bit more complicated than what we found for the conchoid. However, we can extract from (6) an expression for y^2 that is quadratic in x and linear in y. We can use this to reduce (9) to an equation that is cubic in x but linear in y. We then proceed as we did with the conchoid to get a polynomial in x of degree six.

As a consequence of the above analysis, we have shown:

Theorem 5.1. *Suppose α in $\mathbb{C}$ is constructible using a compass and twice-notched straightedge. Then α belongs to a field K that lies in a tower of fields*

$$\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = K$$

for which the index $[K_j : K_{j-1}]$ at each step is 2, 3, 5, or 6. In particular, if $N = [K : \mathbb{Q}]$, then the only primes dividing N are 2, 3, and 5.

Remark. We do not need to include 4 as a possible index in Theorem 5.1. If $[K_{j+1} : K_j] = 4$, then $K_{j+1} = K_j[\beta]$ for β a root of a degree four polynomial in $K_j[x]$. We know that β can be expressed using square and cube roots, so we can replace K_{j+1} with the splitting field for the minimal polynomial for β and insert subfields so that this new field lies in a 2-3-tower over K_j. On the other hand, if $[K_{j+1} : K_j] = 6$ and $K_{j+1} = K_j[\beta]$, then the minimal polynomial for β might not be solvable using radicals, whence K_{j+1} may not exist in a 2-3-5-tower over K_j.

Corollary 5.2. *It is not possible to construct a regular p-gon for $p = 23$ or 29 using only a compass and a twice-notched straightedge.*

Proof. If we can construct the regular p-gon, then we can construct

$$\zeta_p = e^{2\pi i/p},$$

which is the root of an irreducible polynomial of degree $p-1$. By Theorem 5.1, ζ_p lies in a field K of degree N over $\mathbb{Q}$, where the only primes that divide N are 2, 3, and 5. But $\mathbb{Q}[\zeta_p]$ is a subfield of K, so $p-1$ divides N. In particular, for $p = 23$, N must be divisible by 11, and for $p = 29$, N must be divisible by 7. □

A more complete set of p-gons that are not constructible using these tools is shown in Table 1.

6. Points without solid constructions

We will now show that there do indeed exist points that do not have solid constructions but are constructible using a compass and twice-notched straightedge.

Let us choose the pair of conchoids generated by the origin and the line $x = 2$, and intersect them with the circle centered at $(1, 1)$ and with radius $\sqrt{5}$ (see Figure 7). That is, let us choose $a = 2$, $b = c = 1$, and $s = \sqrt{5}$ in (6) and (7). This gives us the polynomial

$$x^6 - 7x^5 + 14x^4 - x^3 - 17x^2 + 18 = 0.$$

We chose the circle so that it would go through the point $(3, 0)$, which is on one of the conchoids, so 3 is a root of this polynomial. Dividing by $(x-3)$, we get

$$f(x) = x^5 - 4x^4 + 2x^3 + 4x^2 + 2x - 6 = 0.$$

It is clear from Eisenstein's criterion that this polynomial is irreducible. Accordingly, the corresponding points of intersection lie in a field whose degree over $\mathbb{Q}$ is divisible by 5, from which we infer that they have no solid construction.

Furthermore, the roots of $f(x)$ cannot be expressed using radicals. To see this, we first note that there are only three points of intersection (Figure 7), so the polynomial $f(x)$ has three real roots and two complex roots. It follows that complex conjugation is an automorphism of the splitting field for $f(x)$ that fixes three roots and transposes the two complex roots; i.e., the Galois group for $f(x)$ includes a transposition. It also, of course, includes a five-cycle, and it is not too hard to show that a subgroup of S_5 that contains both a transposition and five-cycle must be all of S_5. Because S_5 is not solvable, it follows from Abel's classical result that $f(x)$ is not solvable by radicals.

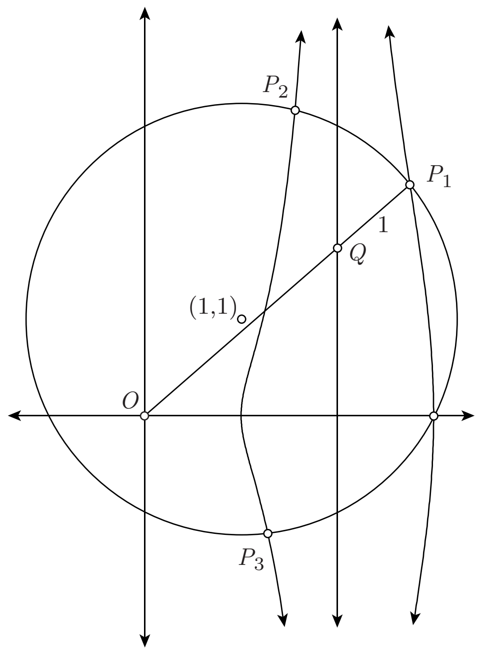

FIGURE 7. Three points P_1, P_2, and P_3, which are constructible using a compass and twice-notched straightedge, but whose x-coordinate is the root of a fifth degree irreducible polynomial that is not solvable by radicals. The point P_1 is found by placing one of the notches Q on the line $y = 2$, the other notch P_1 on the circle centered at $(1, 1)$, and adjusting so that the straightedge goes through the origin.

7. Open questions

This study raises a number of questions that I have been unable to answer.

1. Is it possible to construct a regular 11-gon using a compass and twice-notched straightedge?
2. Is it possible to construct the regular p-gons for $p = 31, 41$, or 61, using only a compass and twice-notched straightedge? What about the 25-gon? (See Table 1.)
3. More generally, are all quintics that are solvable by radicals, also solvable using these tools?

We can answer Question 3 if we can construct the real roots of the following two polynomial equations for any given real a and c:

$$x^5 - a = 0, \tag{10}$$

$$x^5 + 5x^3 - 25x + c = 0. \tag{11}$$

If we can construct the real root of (10), then we can construct fifth roots of lengths. If we can construct the roots of (11), then we can divide any angle α into five equal pieces. To do so, we would choose $c = 2\cos\alpha$. Equation (11) is derived in a fashion similar to the derivation of (3).

To give readers an appreciation for the difficulty of Question 3, we issue them the challenge of coming up with their own proofs of Nicomedes' result. That is, intersect the conchoid of (5) with a class of lines so as to give rise to cubic equations. Then identify the appropriate line that allows one to solve for the cube root of a length. My own attempts have given me a deeper admiration for Nicomedes' accomplishment.

Here are a few more questions:

4. Is it possible to solve all quintics using a compass and twice-notched straightedge?
5. What degree six polynomials are solvable using a compass and twice-notched straightedge?
6. A simpler question: Is it possible to construct a number ω that is the root of an irreducible degree six polynomial not solvable by radicals?

Question 6 would probably make a good undergraduate project.

Finally, let me offer two more possible undergraduate projects: (1) We know that it is possible to construct the regular 7-gon using a compass and twice-notched straightedge. Find such a construction. (2) A *ruler* is a straightedge on which there are rational markings. Prove that a ruler is equivalent to a twice-notched straightedge.

Plane	3, 4, 5, 8, 16, 17, 32, 64
Solid	7, 9, 13, 19, 27, 37, 73, 81, 97
Open	11, 25, 31, 41, 61
No construction	23, 29, 43, 47, 49, 53, 59, 67, 71, 79, 83, 89

TABLE 1. A table of primes and prime powers p^r with $p^r < 100$, identifying whether the p^r-gon has a plane or solid construction, whether it is not constructible using a compass and twice-notched straightedge, or whether the existence of such a construction is an open question.

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